Mixing operators on spaces with weak topology

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Abstract

We prove that a continuous linear operator T on a topological vector space X with weak topology is mixing if and only if the dual operator T' has no finite dimensional invariant subspaces. This result implies the characterization of hypercyclic operators on the space ω due to Herzog and Lemmert and implies the result of Bayart and Matheron, who proved that for any hypercyclic operator T on ω, $T \oplus T$ is also hypercyclic.

1 Introduction

All topological vector spaces in this article *are assumed to be Hausdorff* and are over the field $\mathbb{K}$, being either the field $\mathbb{C}$ of complex numbers or the field $\mathbb{R}$ of real numbers. As usual, $\mathbb{Z}$ is the set of integers and $\mathbb{N}$ is the set of positive integers. If X and Y are vector spaces over the same field $\mathbf{k}$, symbol $L(X, Y)$ stands for the space of $\mathbf{k}$-linear maps from X to Y. If X and Y are topological vector spaces, then $\mathcal{L}(X, Y)$ is the space of continuous linear operators from X to Y. We write $L(X)$ instead of $L(X, X)$, $\mathcal{L}(X)$ instead of $\mathcal{L}(X, X)$ and X' instead of $\mathcal{L}(X, \mathbb{K})$. For each $T \in \mathcal{L}(X)$, the dual operator $T' : X' \to X'$ is defined as usual: $(T'f)(x) = f(Tx)$ for $f \in X'$ and $x \in X$. We say that the topology τ of a topological vector space X *is weak* if τ is exactly the weakest topology making each $f \in Y$ continuous for some linear space Y of linear functionals on X separating points of X. We use symbol ω to denote the product of countably many copies of $\mathbb{K}$. It is easy to see that ω is a separable complete metrizable topological vector space, whose topology is weak.

Let X be a topological vector space and $T \in \mathcal{L}(X)$. A vector $x \in X$ is called a *hypercyclic vector* for T if $\{T^n x : n \in \mathbb{N}\}$ is dense in X and T is called *hypercyclic* if it has a hypercyclic vector. Recall also that T is called *hereditarily hypercyclic* if for each infinite subset A of $\mathbb{N}$, there is $x \in X$ such that $\{T^n x : n \in A\}$ is dense in X. Next, T is called *transitive* if for any non-empty open subsets U and V of X, there is $n \in \mathbb{N}$ for which $T^n(U) \cap V \neq \varnothing$ and T is called *mixing* if for any non-empty open subsets U and V of X, there is $n \in \mathbb{N}$ such that $T^m(U) \cap V \neq \varnothing$ for each $n \geqslant m$. It is well-known and easy to see that any hypercyclic operator (on any topological vector space) is transitive and any hereditarily hypercyclic operator is mixing. If X is complete separable and metrizable, then the converse implications hold: any transitive operator is hypercyclic and any mixing operator is hereditarily hypercyclic. For the proof of these facts as well as for any additional information on the above classes of operators we refer to the book [2] and references therein. Herzog and Lemmert [5] characterized hypercyclic operators on ω.

Theorem HL. *Let $\mathbb{K} = \mathbb{C}$ and $T \in \mathcal{L}(\omega)$. Then T is hypercyclic if and only if the point spectrum $\sigma_p(T')$ of T' is empty.*

Another result concerning hypercyclic operators on ω is due to Bayart and Matheron [1].

Theorem BM. *For any hypercyclic operator $T \in \mathcal{L}(\omega)$, $T \oplus T$ is also hypercyclic.*

We refer to [3, 6] for results on the structure of the set of hypercyclic vectors of operators on ω and to [4, 7] for results on hypercyclicity of operators on Banach spaces endowed with its weak topology. We characterize transitive and mixing operators on spaces with weak topology.

Theorem 1.1. *Let X be a topological vector space, whose topology is weak and $T \in \mathcal{L}(X)$. Then the following conditions are equivalent:*

(1.1.1) *T' has no non-trivial finite dimensional invariant subspaces;*

(1.1.2) *T is transitive;*

(1.1.3) *T is mixing;*

(1.1.4) *for any non-empty open subsets U and V of X, there is $k \in \mathbb{N}$ such that $p(T)(U) \cap V \neq \varnothing$ for any polynomial p of degree $\geqslant k$.*

Since ω is complete, separable, metrizable and carries weak topology, we obtain the following corollary.

Corollary 1.2. *Let $T \in \mathcal{L}(\omega)$. Then the following conditions are equivalent*

(1.2.1) *T' has no non-trivial finite dimensional invariant subspaces;*

(1.2.2) *T is hypercyclic;*

(1.2.3) *T is hereditarily hypercyclic;*

(1.2.4) *for any sequence $\{p_k\}_{k\in\mathbb{N}}$ of polynomials with $\deg p_k \to \infty$, there is $x \in \omega$ such that $\{p_k(T)x : k \in \mathbb{N}\}$ is dense in ω.*

Note that in the case $\mathbb{K} = \mathbb{C}$, T' has no non-trivial finite dimensional invariant subspaces if and only if $\sigma_p(T') = \varnothing$. Moreover, the direct sum of two mixing operators is always mixing. Thus Theorem HL and Theorem BM follow from Theorem 1.1.

Remark 1.3. Chan and Sanders [4] observed that on $(\ell_2)_\sigma$, being ℓ_2 with the weak topology, there is a transitive non-hypercyclic operator. Theorem 1.1 provides a huge supply of such operators. For instance, the backward shift T on ℓ_2 is mixing on $(\ell_2)_\sigma$ since T' has no non-trivial finite dimensional invariant subspaces and T is non-hypercyclic since each its orbit is bounded.

Since each weak topology is determined by the corresponding space of linear functionals, it comes as no surprise that Theorem 1.1 is algebraic in nature. Indeed, we derive it from the following characterization of linear maps without finite dimensional invariant subspaces. The idea of the proof is close to that Herzog and Lemmert [5], although by reasoning on a more abstract level, we were able to get a result, which is simultaneously stronger and more general.

We start by introducing some notation. Let $\mathbf{k}$ be a field. Symbol $\mathcal{P}$ stands for the algebra $\mathbf{k}[t]$ of polynomials in one variable over $\mathbf{k}$, while $\mathcal{R}$ is the field $\mathbf{k}(t)$ of rational functions in one variable over $\mathbf{k}$. Consider the $\mathbf{k}$-linear map

$$M : \mathcal{R} \to \mathcal{R}, \qquad Mf(z) = zf(z).$$

If A is a set and X is a vector space, then $X^{(A)}$ stands for the algebraic direct sum of copies of X labeled by A:

$$X^{(A)} = \bigoplus_{\alpha \in A} \mathcal{R} = \left\{ x \in X^A : \{\alpha \in A : x_\alpha \neq 0\} \text{ is finite} \right\}.$$

Symbol $M^{(A)}$ stands for the linear operator on $\mathcal{R}^{(A)}$, being the direct sum of copies of M labeled by A. That is,

$$M^{(A)} \in L(\mathcal{R}^{(A)}), \quad (M^{(A)}f)_\alpha = Mf_\alpha \quad \text{for each } \alpha \in A.$$

It is easy to see that each $M^{(A)}$ has no non-trivial finite dimensional invariant subspaces. Obviously, the same holds true for each restriction of $M^{(A)}$ to an invariant subspace.

Theorem 1.4. *Let X be a vector space over a field $\mathbf{k}$ and $T \in L(X)$. Then T has no non-trivial finite dimensional invariant subspaces if and only if T is similar to a restriction of some $M^{(A)}$ to an invariant subspace.*

The above theorem is interesting on its own right. It also allows us to prove the following lemma, which is the key ingredient in the proof of Theorem 1.1.

Lemma 1.5. *Let X be a non-trivial vector space over a field $\mathbf{k}$ and $T : X \to X$ be a linear map with no non-trivial finite dimensional invariant subspaces. Then for any finite dimensional subspace L of X, there is $m = m(L) \in \mathbb{N}$ such that $p(T)(L) \cap L = \{0\}$ for each $p \in \mathcal{P}$ with $\deg p \geqslant m$.*

2 Linear maps without finite dimensional invariant subspaces

Throughout this section $\mathbf{k}$ is a field, X is a non-trivial linear space over $\mathbf{k}$ and $T : X \to X$ is a $\mathbf{k}$-linear map. We also denote $\mathcal{P}^* = \mathcal{P} \setminus \{0\}$.

Lemma 2.1. *Let T be a linear operator on a linear space X. Then T has no non-trivial finite dimensional invariant subspaces if and only if $p(T)$ is injective for any non-zero polynomial p.*

Proof. If p is a non-zero polynomial and $p(T)$ is non-injective, then there is non-zero $x \in X$ such that $p(T)x = 0$. Let $k = \deg p$. It is straightforward to verify that $E = \operatorname{span}\{x, Tx, \ldots, T^{k-1}x\}$ is a non-trivial finite dimensional invariant subspace for T. Assume now that T has a non-trivial finite dimensional invariant subspace L and p is the characteristic polynomial of the restriction of T to L. By the Hamilton–Cayley theorem, $p(T)$ vanishes on L. Hence $p(T)$ is non-injective. $\qquad\square$

Definition 2.2. For a linear operator T on a vector space X we say that vectors $x_1, \ldots, x_n$ in X are *T-independent* if for any polynomials $p_1, \ldots, p_n$, the equality $p_1(T)x_1 + \ldots + p_n(T)x_n = 0$ implies $p_j = 0$ for $1 \leqslant j \leqslant n$. Otherwise, we say that $x_1, \ldots, x_n$ are *T-dependent*. A set $A \subset X$ is called *T-independent* if any pairwise different vectors $x_1, \ldots, x_n \in A$ are T-independent.

For a subset A of a vector space X and $T \in L(X)$, we denote

$$E(A, T) = \operatorname{span}\left(\bigcup_{n=0}^{\infty} T^n(A)\right) \quad \text{and} \quad F(A, T) = \bigcup_{p \in \mathcal{P}^*} p(T)^{-1}(E(A, T)). \tag{2.1}$$

Clearly, $E(A, T)$ is the smallest subspace of X, containing A and invariant with respect to T and $F(A, T)$ consists of all $x \in X$ for which

$$q(T)x = \sum_{a \in A} p_a(T)a \tag{2.2}$$

for some $q \in \mathcal{P}^*$ and $p = \{p_a\}_{a \in A} \in \mathcal{P}^{(A)}$. Since $p_a \neq 0$ for finitely many $a \in A$ only, the sum in the above display is finite.

Lemma 2.3. *Let $T \in L(X)$ be a linear operator with no non-trivial finite dimensional invariant subspaces and $A \subset X$ be a T-independent set. Then $F(A, T)$ is a linear subspace of X invariant for T and for every $x \in F(A, T)$, the rational functions $f_{x,a} = \frac{p_a}{q}$ with $p \in \mathcal{P}^{(A)}$ and $q \in \mathcal{P}^*$ satisfying (2.2) are uniquely determined by x and $a \in A$. Moreover, the map*

$$J : F(A, T) \to \mathcal{R}^{(A)}, \quad J_x = \{f_{x,a}\}_{a \in A} \tag{2.3}$$

is linear, injective and satisfies $JTx = M^{(A)}Jx$ for any $x \in F(A, T)$. In particular, the restriction $T_A = T\big|_{F(A,T)} \in L(F(A, T))$ is similar to the restriction of $M^{(A)}$ to the invariant subspace $J(F(A, T))$.

Proof. First, we show that the rational functions $f_{x,a} = \frac{p_a}{q}$ for $a \in A$ are uniquely determined by $x \in F(A, T)$. Assume that $q_1, q_2 \in \mathcal{P}^*$ and $\{p_{1,a}\}_{a \in A}, \{p_{2,a}\}_{a \in A} \in \mathcal{P}^{(A)}$ are such that

$$q_1(T)x = \sum_{a \in A} p_{1,a}(T)a \quad \text{and} \quad q_2(T)x = \sum_{a \in A} p_{2,a}(T)a.$$

Applying $q_2(T)$ to the first equality and $q_1(T)$ to the second, we get

$$(q_1 q_2)(T)x = \sum_{a \in A} (q_2 p_{1,a})(T)a = \sum_{a \in A} (q_1 p_{2,a})(T)a.$$

Since A is T-independent, $q_2 p_{1,a} = q_1 p_{2,a}$ for each $a \in A$. That is, $\frac{p_{1,a}}{q_1} = \frac{p_{2,a}}{q_2}$. Thus the rational functions $f_{x,a} = \frac{p_a}{q}$ for $a \in A$ are uniquely determined by x. It is also clear that the set $\{a \in A : f_{x,a} \neq 0\}$ is finite for each $x \in X$. Thus the formula (2.3) defines a map $J : F(A, T) \to \mathcal{R}^{(A)}$.

Our next step is to show that $F(A, T)$ is a linear subspace of X and that the map J is linear. Let $x, y \in F(A, T)$ and $t, s \in \mathbf{k}$. Pick $q_1, q_2 \in \mathcal{P}^*$ and $\{p_{1,a}\}_{a \in A}, \{p_{2,a}\}_{a \in A} \in \mathcal{P}^{(A)}$ such that

$$q_1(T)x = \sum_{a \in A} p_{1,a}(T)a \quad \text{and} \quad q_2(T)y = \sum_{a \in A} p_{2,a}(T)a.$$

Hence

$$(q_1 q_2)(T)(tx + sy) = \sum_{a \in B} (t p_{1,a} q_2 + s p_{2,a} q_1)(T)a.$$

It follows that $tx + sy \in F(A, T)$ and therefore $F(A, T)$ is a linear subspace of X. Moreover, by definition of the rational functions $f_{x,a}$, we have $f_{x,a} = \frac{p_{1,a}}{q_1}$, $f_{y,a} = \frac{p_{2,a}}{q_2}$ and

$$f_{tx+sy,a} = \frac{t p_{1,a} q_2 + s p_{2,a} q_1}{q_1 q_2} = t f_{x,a} + s f_{y,a} \quad \text{for any } a \in A,$$

which proves linearity of J. Since $Jx = 0$ if and only if $q(T)x = 0$ for some $q \in \mathcal{P}^*$, Lemma 2.1 implies that $\ker J = \{0\}$. That is, J is injective.

Now let us show that $F(A, T)$ is invariant for T and that $JTx = M^{(A)}Jx$ for any $x \in F(A, T)$. Let $x \in F(A, T)$ and $q \in \mathcal{P}^*$ and $\{p_a\}_{a \in A} \in \mathcal{P}^{(A)}$ be such that

$$q(T)x = \sum_{a \in A} p_a(T)a. \quad \text{Then} \quad q(T)(Tx) = \sum_{a \in B} p_{1,a}(T)a, \quad \text{where } p_{1,a}(z) = z p_a(z).$$

Hence $Tx \in F(A, T)$ and therefore $F(A, T)$ is invariant for T. Moreover, $f_{Tx,a} = \frac{p_{1,a}}{q} = M f_{x,a}$ for any $x \in F(A, T)$ and $a \in A$. That is, $JTx = M^{(A)}Jx$ for any $x \in F(A, T)$. Since J is injective it is a linear isomorphism of $F(A, T)$ and $Y = J(F(A, T))$. Then the equality $JTx = M^{(A)}Jx$ for $x \in F(A, T)$ implies that Y is invariant for $M^{(A)}$ and that T_A is similar to $M^{(A)}\big|_Y$ with the linear map J providing the similarity. $\square$

2.1 Proof of Theorem 1.4

Let $T \in L(X)$ be without non-trivial finite dimensional invariant subspaces. A standard application of the Zorn lemma allows us to take a maximal by inclusion T-independent subset A of X. From the definition of the spaces $F(B, T)$ it follows that if $B \subset X$ is T-independent and $x \in X \backslash F(B, T)$, then $B \cup \{x\}$ is also T independent. Thus maximality of A implies that $X = F(A, T)$. By Lemma 2.3, $T = T_A$ is similar to a restriction of $M^{(A)}$ to an invariant subspace.

2.2 Proof of Lemma 1.5

Let $A \subset L$ be a linear basis of L. Since L is finite dimensional, A is finite. Pick a maximal by inclusion T-independent subset B of A (since A is finite, we do not need the Zorn lemma to do that). Now let $F = F(B,T)$ be the subspace of X defined in (2.1). Since B is a maximal T-independent subset of a basis of L, $L \subseteq F$. By Lemma 2.3, F is invariant for T. Thus we can without loss of generality assume that $X = F$. Then by Lemma 2.3, we can assume that T is a restriction of $M^{(B)}$ to an invariant subspace. Since extending T beyond X is not going to change the spaces $L \cap p(T)(L)$, we can assume that $T = M^{(B)}$. Since B is finite, without loss of generality, $X = \mathcal{R}^n$ and $T = M \oplus \ldots \oplus M$, where $n \in \mathbb{N}$.

Consider the degree function $\deg : \mathcal{R} \to \mathbb{Z} \cup \{-\infty\}$. We set $\deg(0) = -\infty$ and let $\deg(p/q) = \deg p - \deg q$, where p and q are non-zero polynomials and the degrees in the right hand side are the conventional degrees of polynomials. Clearly this function is well-defined and is a grading on $\mathcal{R}$. That is,

(g1) $\deg(f_1 f_2) = \deg(f_1) + \deg(f_2)$ and $\deg(f_1 + f_2) \leqslant \max\{\deg f_1, \deg f_2\}$ for any $f_1, f_2 \in \mathcal{R}$;

(g2) if $f_1, f_2 \in \mathcal{R}$ and $\deg f_1 \neq \deg f_2$, then $\deg(f_1 + f_2) = \max\{\deg f_1, \deg f_2\}$.

By (g1), $\deg(Mf) = 1 + \deg f$ for each $f \in \mathcal{R}$. For $f \in X = \mathcal{R}^n$, we write

$$\delta(f) = \max_{1 \leqslant j \leqslant n} \deg f_j.$$

Clearly $\delta(0) = -\infty$ and $\delta(f) \in \mathbb{Z}$ for each $f \in X \setminus \{0\}$. Let also

$$\Delta^+ = \sup_{f \in L} \delta(f) \quad \text{and} \quad \Delta^- = \inf_{f \in L \setminus \{0\}} \delta(f).$$

Then Δ^+ and Δ^- are finite. Indeed, assume that either $\Delta^+ = +\infty$ or $\Delta^- = -\infty$. Then there exists a sequence $\{u_l\}_{l \in \mathbb{N}}$ in $L \setminus \{0\}$ such that $\{\delta(u_l)\}_{l \in \mathbb{N}}$ is strictly monotonic. For each l we can pick $j(l) \in \{1, \ldots, n\}$ such that $\delta(u_l) = \deg(u_l)_{j(l)}$. Then there is $\nu \in \{1, \ldots, n\}$ for which the set $B_\nu = \{l \in \mathbb{N} : j(l) = \nu\}$ is infinite. It follows that the degrees of $(u_l)_\nu$ for $l \in B_\nu$ are pairwise different. Property (g2) of the degree function implies that the rational functions $(u_l)_\nu$ for $l \in B_\nu$ are linearly independent. Hence the infinite set $\{u_l : l \in B_\nu\}$ is linearly independent in X, which is impossible since all u_l belong to the finite dimensional space L. Thus Δ^+ and Δ^- are finite.

Now let $p \in \mathcal{P}^*$ and $d = \deg p$. By (g1) and the equality $(Tf)_j = Mf_j$, $\delta(p(T)f) = \delta(f) + d$ for each $f \in X$. Therefore, $\inf\{\delta(f) : f \in p(T)(L) \setminus \{0\}\} = \Delta^- + d$. In particular, if $d > \Delta^+ - \Delta^-$, then

$$\inf_{f \in p(T)(L) \setminus \{0\}} \delta(f) = \Delta^- + d > \Delta^+ = \sup_{f \in L} \delta(f).$$

Thus $\delta(u) > \delta(v)$ for any non-zero $u \in P(T)(L)$ and $v \in L$, which implies that $p(T)(L) \cap L = \{0\}$ whenever $\deg p > \Delta^+ - \Delta^-$. Thus the number $m = \Delta^+ - \Delta^- + 1$ satisfies the desired condition. The proof of Lemma 1.5 is complete.

3 Proof of Theorem 1.1

The implications $(1.1.4) \implies (1.1.3) \implies (1.1.2)$ are trivial. Assume that T is transitive and T' has a non-trivial finite dimensional invariant subspace. Then T has a non-trivial closed invariant subspace of finite codimension. Passing to the quotient by this subspace, we obtain a transitive operator on a finite dimensional topological vector space. Since there is only one Hausdorff vector space topology on a finite dimensional space, we arrive to a transitive operator on a finite dimensional Banach space. Since transitivity and hypercyclicity for operators on separable Banach spaces are equivalent, we obtain a hypercyclic operator on a finite dimensional Banach space. On the other

hand, it is well known that such operators do not exist, see, for instance, [8]. Thus (1.1.2) implies (1.1.1). It remains to show that (1.1.1) implies (1.1.4).

Assume that (1.1.1) is satisfied and (1.1.4) fails. Then there exist non-empty open subsets U and V of X and a sequence $\{p_l\}_{l \in \mathbb{N}}$ of polynomials such that $\deg p_l \to \infty$ and $p_l(T)(U) \cap V = \varnothing$ for each $l \in \mathbb{N}$. Since X carries weak topology, there exist two finite linearly independent sets $\{f_1, \ldots, f_n\}$ and $\{g_1, \ldots, g_m\}$ in X' and two vectors $(a_1, \ldots, a_n) \in \mathbb{K}^n$ and $(b_1, \ldots, b_m) \in \mathbb{K}^m$ such that $U_0 \subseteq U$ and $V_0 \subseteq V$, where

$$U_0 = \{u \in X : f_k(u) = a_k \ \text{ for } \ 1 \leqslant k \leqslant n\} \ \text{ and } \ V_0 = \{u \in X : g_j(u) = b_j \ \text{ for } \ 1 \leqslant j \leqslant m\}.$$

Let $L = \operatorname{span}\{f_1, \ldots, f_n, g_1, \ldots, g_m\}$. Since T' has no non-trivial finite dimensional invariant subspaces, by Lemma 1.5, $p_l(T')(L) \cap L = \{0\}$ for any sufficiently large l. For such an l, the equality $p_l(T')(L) \cap L = \{0\}$ together with the injectivity of $p_l(T')$, provided by Lemma 2.1, and the definition of L imply that the vectors $p_l(T')g_1, \ldots, p_l(T')g_m, f_1, \ldots, f_n$ are linearly independent. Hence there exists $u \in X$ such that

$$p_l(T')g_j(u) = b_j \ \text{ for } \ 1 \leqslant j \leqslant m \ \text{ and } \ f_k(u) = a_k \ \text{ for } 1 \leqslant k \leqslant n.$$

Since $p_l(T')g_j(u) = g_j(p_l(T)u)$, the last display implies that $u \in U_0 \subseteq U$ and $p_l(T)u \in V_0 \subseteq V$. Hence $p_l(T)(U) \cap V$ contains $p_l(T)u$ and therefore is non-empty. This contradiction completes the proof of Theorem 1.1.

Remark 3.1. The only place in the proof of Theorem 1.1, where we used the nature of the underlying field, is the reference to the absence of transitive operators on non-trivial finite dimensional spaces. Thus Theorem 1.1 extends to topological vector spaces with weak topology over any topological field $\mathbf{k}$ provided there are no transitive operators on non-trivial finite dimensional topological vector spaces over $\mathbf{k}$.

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